

SINGULARITIES OF GENERIC LINKAGE OF ALGEBRAIC VARIETIES

WENBO NIU

ABSTRACT. Let Y be a generic link of a subvariety X of a nonsingular variety A . By using resolution of singularities and multiplier ideal sheaves we are able to produce a machinery to study the singularities of Y . For example, we give a criterion when Y has rational singularities. Furthermore we study linkage from the viewpoint of pairs so that current singularities theories can be applied. As a consequence, we show that log canonical threshold increases and log canonical pairs are preserved in generic linkage.

1. INTRODUCTION

Let A be either a nonsingular affine variety or a projective space \mathbb{P}^n over the field of complex number \mathbb{C} . Two closed subvarieties of A are said to be geometrically linked if the union of them is a complete intersection in A . Let us fix a subvariety X of A of codimension c for now. By choosing c equations carefully among the defining equations of X we define a complete intersection V and obtain a subscheme Y as the closure of the complement of X in V . Then Y is geometrically linked to X via V , i.e., $Y \cup X = V$ and Y has no common irreducible components with X . If such complete intersection V is chosen to be as general as possible, then Y is called a generic link of X .

The study of linkage, or the theory of liaison, of algebraic varieties can be tracked back to hundreds of years ago as a classical algebraic geometry topic. The recent work in this area was initiated by Peskine and Szpiro [PS74]. After that, linkage has attracted considerable attention and has been developed widely and deeply from both geometric and algebraic point of views.

Let us state the construction of a generic link more concretely to make our introduction clear. Assume further that X is defined by an ideal I_X generated by equations f_1, \dots, f_t . The essential point is to create a general complete intersection V by using the equations f_i 's. There are three situations in current research to do this, which we list as follows.

Situation A. Suppose $A = \text{Spec } R$ is affine. A complete intersection V is defined by the equations $\alpha_i = U_{i,1}f_1 + \dots + U_{i,t}f_t$, for $i = 1, \dots, c$, where $U_{i,j}$'s are variables. V is actually defined in an extended space $B = \text{Spec } R[U_{i,j}]$.

Situation B. Suppose $A = \text{Spec } R$ is affine. A complete intersection V is defined by the equations $\alpha_i = a_{i,1}f_1 + \dots + a_{i,t}f_t$, for $i = 1, \dots, c$, where $a_{i,j}$'s are general scalars in \mathbb{C} .

Situation C. Suppose $A = \mathbb{P}^n$ and the homogeneous equations f_i 's of X have degrees $d_1 \geq d_2 \geq \dots \geq d_t$. A complete intersection V is defined by choosing general equations α_i from $H^0(\mathbb{P}^n, I_X(d_i))$, for $i = 1, \dots, c$.

In each of the three situations, once having a complete intersection V in hand, a generic link can be obtained by algebraic construction. Specifically, a generic link Y of X can be defined by an ideal $I_Y := (I_V : I_X B)$ in Situation A and by $I_Y := (I_V : I_X)$ in Situation B and C.

2010 *Mathematics Subject Classification.* 13C40, 14M06.

Key words and phrases. Generic linkage, singularity, log canonical pair, multiplier ideal sheaf.

The terminology of generic linkage usually refers to Situation A, in which X is extended to the space B and then make linkage construction there. The theory based on this setting has been founded and developed deeply by Huneke and Ulrich in the last three decades, mainly from algebraic side of the story (e.g. [HU88], [HU87]). Situation C was studied from the geometric side. The book of Miglior [Mig98] gives a outline of the theory along this direction. It is worth mentioning that the way to construct a generic link in Situation C is quite classical and a typical application can be found in the work of Betram, Ein and Lazarsfeld [BEL91] on the Castelnuovo-Mumford regularity bound for a smooth projective variety. Situation B can be considered as a specialization of Situation A as well as a local version of Situation C. As an important technique it has been used in recent studies of singularities, for instance, in the work of deFernex and Roi [dFD12] and the work of Ein and Mustata [EM09].

The usage of linkage has a long history in algebraic geometry. With a lot of efforts of mathematicians many features of linkage are well understood now. However, in contrast to the quick and deep development of singularity theories in the past decades, much less has been known about the singularities in generic linkage. A few special examples have drawn attention recently. One important case, which also serves as a guideline of this paper, was observed by Bernd Ulrich that if X is a local complete intersection and has rational singularities then a generic link of X has rational singularities. But the machinery behind Ulrich's observation seems quite mysterious. Meanwhile, experience gained from research in the past gives the intuition that the singularities of a generic link seem to be worse than X itself.

We are trying to make two developments on generic linkage in this paper. First, by using resolution of singularities we provide a machinery to study the singularities of generic linkage. Specifically, we give a concrete way to resolve singularities of a generic link. Second, we study generic linkage as pairs, which makes applications of current singularity theories possible. Consequently, we are able to give a complete answer to the question when a generic link has rational singularities, and reveal the rationale behind Ulrich's observation. It is interesting that multiplier ideal sheaves naturally come into play and turns out to be an efficient tool in this area. Also once we look at generic linkage from viewpoint of pairs, it is then quite surprising that singularities would not become worse as one expected.

We conclude this introduction by briefly stating the organization of this paper. We take Situation A as our framework, following the footprints of Huneke and Ulrich. Main theorems (cf. Theorem 3.3 and Theorem 3.8) are stated and proved in this situation in Section 3. We also give related results in Situation B and C (cf. Theorem 3.15 and Theorem 4.1). In particular Section 4 is devoted to the case of projective varieties.

Acknowledgement. Special thanks are due to professor Bernd Ulrich who introduced the author to this subject and spent his valuable time on discussion. This paper would not be possible without his generous help and encouragement. The author also would like to thank professor Lawrence Ein for his insightful knowledge and inspiring suggestions which enrich the paper. The author's thanks also goes to professor Joseph Lipman for his patient reading and kind suggestions.

2. GENERIC LINKAGE, SINGULARITIES AND MULTIPLIER IDEAL SHEAVES

Throughout this paper, we work over the field $k := \mathbb{C}$. By a variety we mean a reduced irreducible scheme of finite type over k . A subscheme is always assumed to be closed. We shall briefly review basic facts about linkage, singularities and multiplier ideal sheaves in this section.

Let us start with a definition of linkage, more precisely, a definition of geometrical linkage, which is the main object of this paper.

Definition 2.1. Let A be either a nonsingular affine variety over k or a projective space \mathbb{P}_k^n . Let X and Y be two subschemes of A . We say that X and Y are *geometrically linked* if $X \cup Y$ is a complete intersection V in A and X and Y are equidimensional, no embedded components and have no common irreducible components. We also say Y is linked to X via V , or vice versa.

Suppose Y is geometrically linked to X via V in A as defined above. Let ω_Y be a dualizing sheaf of Y (in this paper, we use dualizing sheaf and canonical sheaf interchangeably). One important fact is that there is an isomorphism $\omega_Y \simeq \mathcal{I}_X \cdot \mathcal{O}_Y \otimes \omega_V$, where \mathcal{I}_X is the defining ideal sheaf of X in A and ω_V is a dualizing sheaf of V . If A is an affine nonsingular variety, then one can deduce that

$$(2.1.1) \quad \omega_Y \simeq \mathcal{I}_X \cdot \mathcal{O}_Y \otimes \omega_A,$$

where ω_A is a dualizing sheaf of A . If A is a projective space \mathbb{P}_k^n and V is cut out by homogeneous equations of degrees $d_1 \geq d_2 \geq \cdots \geq d_c$, where $c = \text{codim } V$, then one has

$$(2.1.2) \quad \omega_Y \simeq \mathcal{I}_X \cdot \mathcal{O}_Y \otimes \omega_A(d_1 + \cdots + d_c).$$

It is also well-known that Y is Cohen-Macaulay if and only if X is Cohen-Macaulay. The modern approach to study linkage goes back to Peskine and Szpiro [PS74], and we refer to their work for more general theory of linkage.

Definition 2.1 suggests that the study of linkage involves an ambient space containing the varieties concerned. Thus it is natural to consider a variety and its ambient space as a pair. Besides, the concept of pairs is a successful approach to study singularities of varieties, which is the main motivation of this paper.

Definition 2.2. A *pair* (A, cX) contains a nonsingular variety A over k , a subscheme X and a nonnegative real number c . If $A = \text{Spec } R$ is affine and X is defined by an ideal I_X , then we say the pair (A, cX) is an affine pair and also use an alternative notation (R, I_X^c) .

Now we give the definition of generic linkage mentioned in Situation A in Introduction, which is the main framework of this paper.

Definition 2.3. Let $(A_k, cX_k) = (R_k, I_{X_k}^c)$ be an affine pair such that X_k is reduced equidimensional and $c = \text{codim}_{A_k} X_k$. We construct a generic link of X_k as follows. Fix a generating set (f_1, \dots, f_t) of I_{X_k} . Let $(U_{ij}), 1 \leq i \leq c, 1 \leq j \leq t$, be a $c \times t$ matrix of variables. Set $R := R_k[U_{ij}]$ and $I_X := I_{X_k} \cdot R_k[U_{ij}]$ and define $A = \text{Spec } R$ and $X = \text{Spec } R/I_X$. Notice that I_X is still generated by (f_1, \dots, f_t) in R . We define a complete intersection V inside A by an ideal

$$I_V := (\alpha_1, \dots, \alpha_c) = (U_{i,j}) \cdot (f_1, \dots, f_t)^T,$$

that is

$$\alpha_i := U_{i,1}f_1 + U_{i,2}f_2 + \cdots + U_{i,t}f_t, \quad \text{for } 1 \leq i \leq c.$$

Then a generic link Y to X_k via V is defined by an ideal $I_Y := (I_V : I_X)$.

Remark 2.4. In the definition above, the subscheme V is a complete intersection in A [Hoc73]. The subschemes Y and X of A are geometrically linked so that $I_Y = (I_V : I_X)$, $I_X = (I_V : I_Y)$ and $I_V = I_X \cap I_Y$ and therefore Y is equidimensional without embedded components and has no common components with X [HU85, 2.1, 2.5]. Furthermore Y is actually integral [HU85, 2.6]. In the sequel we only need the fact that Y is reduced equidimensional. It is from (2.1.1) that a dualizing sheaf ω_Y of Y is $\omega_Y \simeq I_X \cdot \mathcal{O}_Y \otimes \omega_A$, where ω_A is a dualizing sheaf of A .

Let A be a nonsingular variety over k and $\{X_i\}$, $i = 1, \dots, m$, are m subschemes of A . By Hironaka's resolution of singularities, there is a birational projective morphism $f : A' \rightarrow A$ such that A' is a nonsingular variety, $f^{-1}(X_i) = \sum a_{i,j} E_j$ for $i = 1, \dots, m$, where $a_{i,j}$'s are nonnegative integers and E_j 's are prime divisors of A' such that the union of E_j 's with the exceptional locus $\text{Exc}(f)$ is a simple normal crossing divisor. The morphism f is called a *log resolution* of $(A, \sum_i X_i)$.

Let (A, cX) be a pair. Take a log resolution $f : A' \rightarrow A$ of (A, X) such that $f^{-1}(X) = \sum_{i=1}^s a_i E_i$ and the relative canonical divisor $K_{A'/A} = \sum_{i=1}^s k_i E_i$. We say that the pair (A, cX) is *log canonical* if $k_i - c \cdot a_i \geq -1$ for all i . The *multiplier ideal sheaf* $\mathcal{J}(A, cX)$ associated to the pair (A, cX) is defined to be

$$\mathcal{J}(A, cX) := f_* \mathcal{O}_{A'}(K_{A'/A} - \lfloor c \sum a_i E_i \rfloor),$$

where $\lfloor c \sum a_i E_i \rfloor$ is the round down of the \mathbb{Q} -divisor $c \sum a_i E_i$.

Lemma 2.5. *Let A be a nonsingular variety over k and $X \subset A$ be a reduced equidimensional subscheme of codimension c defined by an ideal sheaf \mathcal{I}_X . Let $\mathcal{J}(A, cX)$ be the multiplier ideal sheaf associated to the pair (A, cX) . Then $\mathcal{J}(A, cX) \subseteq \mathcal{I}_X$.*

Proof. Since at each generic point $p \in A$ of X one has $\mathcal{J}(A, cX)_p = \mathcal{I}_{X,p}$ and \mathcal{I}_X is radical the result is then clear. \square

The following lemma is due to Lawrence Ein, which gives a criterion to compare multiplier ideal sheaves with ideal sheaves.

Ein's Lemma. *Let A be a nonsingular variety and $X \subset A$ be a reduced equidimensional subscheme of codimension c defined by an ideal sheaf \mathcal{I}_X . Then $\mathcal{J}(A, cX) = \mathcal{I}_X$ if and only if $\mathcal{J}(A, (c-1)X) = \mathcal{O}_A$. In particular if the pair (A, cX) is log canonical then $\mathcal{J}(A, cX) = \mathcal{I}_X$.*

Proof. The inclusion $\mathcal{J}(A, cX) \subseteq \mathcal{I}_X$ is from Lemma 2.5. It then suffices to show that $\mathcal{I}_X \subseteq \mathcal{J}(A, cX)$ if and only if $\mathcal{J}(A, (c-1)X)$ is trivial. If $c = 1$, then there is nothing to prove. So in the sequel we assume $c > 1$. We shall follow notation and terminologies in [EM09, Section 7].

The inclusion $\mathcal{I}_X \subseteq \mathcal{J}(A, cX)$ is true if and only if for any prime divisor E over X we have an inequality $\text{ord}_E \mathcal{I}_X + \text{ord}_E K_{-/A} - c \cdot \text{ord}_E \mathcal{I}_X \geq 0$. This is equivalent to the inequality $\text{ord}_E K_{-/A} - (c-1) \cdot \text{ord}_E \mathcal{I}_X \geq 0$, which is equivalent to $\mathcal{J}(A, (c-1)X)$ is trivial.

Now suppose (A, cX) is log canonical. This means that for any prime divisor E over X we have $\text{ord}_E K_{-/A} - c \cdot \text{ord}_E \mathcal{I}_X \geq -1$. If the center of E is outside X then $\text{ord}_E \mathcal{I}_X = 0$ and $\text{ord}_E K_{-/A} \geq 0$ since A is nonsingular. And thus $\text{ord}_E K_{-/A} - c \cdot \text{ord}_E \mathcal{I}_X \geq -\text{ord}_E \mathcal{I}_X$. On the other hand, if the center of E is inside X , then $\text{ord}_E \mathcal{I}_X \geq 1$ and again we have $\text{ord}_E K_{-/A} - c \cdot \text{ord}_E \mathcal{I}_X \geq -1 \geq -\text{ord}_E \mathcal{I}_X$. Hence in any case we have $\text{ord}_E K_{-/A} - c \cdot \text{ord}_E \mathcal{I}_X \geq -\text{ord}_E \mathcal{I}_X$, which implies, as showed above, that $\mathcal{J}(A, (c-1)X)$ is trivial. \square

Definition 2.6. Let A be a nonsingular variety over k and X a reduced subscheme of A . A morphism $\varphi_A : \bar{A} \rightarrow A$ is a *factorizing resolution* of X inside A if the following hold:

- (1) φ_A is an isomorphism at the generic point of every irreducible component of X . In particular, the strict transform \bar{X} of X is defined.
- (2) The morphism φ_A and $\varphi_X := \varphi_A|_{\bar{X}}$ are resolution of singularities of A and X , respectively, and the union of \bar{X} with the exceptional locus $\text{Exc}(\varphi_A)$ has simple normal crossings.

- (3) If I_X and $I_{\overline{X}}$ are the defining ideals of X and \overline{X} in A and \overline{A} , respectively, then there exists an effective divisor G on \overline{A} such that

$$I_X \cdot \mathcal{O}_{\overline{A}} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-G).$$

The divisor G is supported on $\text{Exc}(\varphi_A)$ and hence has normal crossing with \overline{X} .

Remark 2.7. The above definition is borrowed from [EIM11, Definition 2.10]. The existence of a factorizing resolution is proved in [BVU03, Theorem 1.2]. In addition, we can assume that the morphism φ_A is isomorphic over the open set $A \setminus X$.

Let us conclude this section by briefly reviewing the definition of rational singularities and Grauert-Riemenschneider canonical sheaves. Let X be a reduced equidimensional scheme of finite type over k . Let $f : X' \rightarrow X$ be a resolution of singularities of X . Then the *Grauert-Riemenschneider canonical sheaf* ω_X^{GR} of X is defined to be $\omega_X^{GR} := f_*\omega_{X'}$, where $\omega_{X'}$ is the canonical sheaf of X' . It turns out that the sheaf ω_X^{GR} is independent on the choice of the resolution of singularities f (cf. [Laz04a]). Furthermore ω_X^{GR} is canonically a subsheaf of ω_X , a dualizing sheaf of X , via a trace map $\text{tr} : \omega_X^{GR} \hookrightarrow \omega_X$. Recall that X has *rational singularities* if $f_*\mathcal{O}_{X'} = \mathcal{O}_X$ and $R^if_*\mathcal{O}_{X'} = 0$ for $i > 0$. It is well-known that X has rational singularities if and only if X is Cohen-Macaulay and $\omega_X^{GR} = \omega_X$ (cf. [Kol97]).

3. GENERIC LINKAGE OF AFFINE VARIETIES

In this section, we study the singularities of generic linkages under the framework of Huneke-Ulrich as Definition 2.3. The basic idea is to produce a “user-friendly” resolution of singularities of a generic link, which is the main machinery used throughout this section. The first key observation we made is that the Grauert-Riemenschneider canonical sheaf of a generic link can be expressed as a multiplier ideal sheaf. The second one is that we give an estimation on log canonical thresholds of a generic link. Many results then follow easily from these two observations. At the end of the section, we specialize our result to a Zariski open set of a scalar matrices space, which takes care of Situation B.

Proposition 3.1. *With notation as in Definition 2.3 let Y be a generic link to a pair (A_k, cX_k) . Then*

$$\omega_Y^{GR} \simeq \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A,$$

where $\mathcal{I}(A, cX)$ is the multiplier ideal sheaf associated to the pair (A, cX) .

Proof. Let $\varphi_k : \overline{A}_k \rightarrow A_k$ be a factorizing resolution of singularities of X_k inside A_k , so that $I_{X_k} \cdot \mathcal{O}_{\overline{A}_k} = I_{\overline{X}_k} \cdot \mathcal{O}_{\overline{A}_k}(-G_k)$ where \overline{X}_k is the strict transformation of X_k , G_k is an effective divisor supported on the exceptional locus of φ_k , and furthermore \overline{X}_k and the exceptional locus of φ_k are simple normal crossings. The morphism φ_k can be assumed to be an isomorphism over the open set $A_k \setminus X_k$ (cf. Definition 2.6 and Remark 2.7).

By tensoring $k[U_{ij}]$ to the factorizing resolution φ_k , we obtain a factorizing resolution of singularities of X inside A as

$$\varphi : \overline{A} \rightarrow A,$$

such that $I_X \cdot \mathcal{O}_{\overline{A}} = I_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-G)$, where \overline{X} is the strict transform of X , G is an effective divisor supported on the exceptional locus of φ , and \overline{X} and exceptional locus of φ are simple normal crossings. Notice that by the construction, we actually have $\overline{A} = \overline{A}_k \otimes_k \text{Spec } k[U_{ij}]$, $\overline{X} = \overline{X}_k \otimes_k \text{Spec } k[U_{ij}]$ and $G = G_k \otimes_k \text{Spec } k[U_{ij}]$.

Claim 3.1.1. The ideal sheaf $I_V \cdot \mathcal{O}_{\bar{A}}$ has a decomposition as

$$I_V \cdot \mathcal{O}_{\bar{A}} = I_{\bar{V}} \cdot \mathcal{O}_{\bar{A}}(-G)$$

where $I_{\bar{V}}$ is an ideal sheaf on \bar{A} and is a local complete intersection.

Proof of Claim 3.1.1. The question is local. Recall that $\varphi_k : \bar{A}_k \rightarrow A_k$ is the factorizing resolution of singularities of X_k inside A_k . Let $\bar{U}_k = \text{Spec } \bar{R}_k$ be an affine open set of \bar{A}_k such that the effective divisor G_k is defined by an equation $g \in \bar{R}_k$ and we have a decomposition $I_{X_k} \cdot \bar{R}_k = I_{\bar{X}_k} \cdot (g)$ on \bar{U}_k . Now since $I_{X_k} \cdot \bar{R}_k = (f_1, \dots, f_t) \cdot \bar{R}_k$ we can write $f_i = \bar{f}_i g$ where $\bar{f}_i \in \bar{R}_k$ for $i = 1, \dots, t$ so that $I_{\bar{X}_k} = (\bar{f}_1, \dots, \bar{f}_t)$.

Recall that the factoring resolution $\varphi : \bar{A} \rightarrow A$ is obtained by tensoring $\text{Spec } k[U_{ij}]$ to the factoring resolution φ_k . Write $\bar{R} = \bar{R}_k \otimes k[U_{ij}]$ which is a faithfully flat ring extension of \bar{R}_k and then $\bar{U} = \bar{U}_k \otimes \text{Spec } k[U_{ij}] = \text{Spec } \bar{R}$ is an affine open set of \bar{A} . Notice that on \bar{U} the ideal $I_{\bar{X}} = I_{\bar{X}_k} \cdot \bar{R}$ and the effective divisor G is still generated by the equation g . Recall that the ideal $I_V = (\alpha_1, \dots, \alpha_c)$, where $\alpha_i = U_{i,1}f_1 + U_{i,2}f_2 + \dots + U_{i,t}f_t$. Thus if write $\bar{\alpha}_i = U_{i,1}\bar{f}_1 + U_{i,2}\bar{f}_2 + \dots + U_{i,t}\bar{f}_t$ and set $I_{\bar{V}} = (\bar{\alpha}_1, \dots, \bar{\alpha}_c)$, then $I_{\bar{V}}$ is a complete intersection on \bar{U} and we have a decomposition $I_V \cdot \bar{R} = I_{\bar{V}} \cdot (g)$ on \bar{U} , which finishes the proof of Claim 3.1.1.

Now let $\mu : \tilde{A} \rightarrow \bar{A}$ be the blowing-up of \bar{A} along \bar{X} such that $I_{\bar{X}} \cdot \mathcal{O}_{\tilde{A}} = \mathcal{O}_{\tilde{A}}(-T)$, where T is an exceptional divisor of μ . Denote by $\psi = (\phi \circ \mu) : \tilde{A} \rightarrow A$. Notice that the supports of divisors T , $\mu^*(G)$ and the exceptional locus of ψ are simple normal crossings. We write $K_{\tilde{A}/A}$ to be the relative canonical divisor of the morphism ψ .

Claim 3.1.2. We have the following statements.

- (1) The ideal sheaf $I_{\bar{V}} \cdot \mathcal{O}_{\tilde{A}}$ can be decomposed as

$$I_{\bar{V}} \cdot \mathcal{O}_{\tilde{A}} = I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(-T).$$

where $I_{\tilde{V}}$ is an ideal sheaf on \tilde{A} and defines a local complete intersection \tilde{V} of codimension c .

- (2) The scheme \tilde{V} is nonsingular and irreducible and its dualizing sheaf is

$$\omega_{\tilde{V}} \simeq \mathcal{O}_{\tilde{V}}(K_{\tilde{A}/A} - c(T + \mu^*G)) \otimes \psi^*\omega_A,$$

where ω_A is a dualizing sheaf of A .

- (3) The scheme \tilde{V} is the strict transform of V via ψ .

Proof of Claim 3.1.2. We work locally on affine open sets as in the proof of Claim 3.1.1. Assume that $\bar{A}_k = \text{Spec } \bar{R}_k$ and $\bar{A} = \text{Spec } \bar{R}$, where $\bar{R} = \bar{R}_k \otimes k[U_{ij}]$. Recall that, as we showed in the proof of Claim 3.1.1, the ideal $I_{\bar{X}} = (\bar{f}_1, \dots, \bar{f}_t) \cdot \bar{R}$ where for each i the generator \bar{f}_i is in the ring \bar{R}_k , and that the complete intersection $I_{\bar{V}} = (\bar{\alpha}_1, \dots, \bar{\alpha}_c)$, where $\bar{\alpha}_i = U_{i,1}\bar{f}_1 + U_{i,2}\bar{f}_2 + \dots + U_{i,t}\bar{f}_t$ for $i = 1, \dots, c$. Now take a canonical affine cover of \tilde{A} , say $U = \text{Spec } \bar{R}[\bar{f}_2/\bar{f}_1, \dots, \bar{f}_t/\bar{f}_1]$ such that the exceptional divisor T is given by the element \bar{f}_1 on U . For $i = 1, \dots, c$ write $\tilde{\alpha}_i = U_{i,1} + U_{i,2}\bar{f}_2/\bar{f}_1 + \dots + U_{i,t}\bar{f}_t/\bar{f}_1$ and set $I_{\tilde{V}} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_c)$. Then on the open set U we have $I_{\bar{V}} \cdot \mathcal{O}_U = I_{\tilde{V}} \cdot (f_1)$ and $I_{\tilde{V}}$ defines an irreducible nonsingular variety of \tilde{V} on U , which prove the statement (1) and the first part of the statement (2) in the claim.

Next we compute the dualizing sheaf of \tilde{V} . Notice that we have $I_V \cdot \mathcal{O}_{\tilde{A}} = I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(-T - \mu^*G)$. Since I_V is a complete intersection in A of codimension c we have a surjective morphism

$\oplus^c \mathcal{O}_A \longrightarrow I_V \longrightarrow 0$, which induces on \tilde{A} a surjective morphism

$$\bigoplus^c \mathcal{O}_{\tilde{A}}(T + \mu^* G) \longrightarrow I_{\tilde{V}} \longrightarrow 0.$$

Thus it is clear that the determinant of the normal bundle of \tilde{V} inside \tilde{A} is

$$\det N_{\tilde{V}/\tilde{A}} = \mathcal{O}_{\tilde{V}}(-c(T + \mu^* G)).$$

Then by the adjunction formula, we have

$$\omega_{\tilde{V}} \simeq \omega_{\tilde{A}} \otimes \det N_{\tilde{V}/\tilde{A}} = \mathcal{O}_{\tilde{V}}(K_{\tilde{A}/A} - c(T + \mu^* G)) \otimes \psi^* \omega_A,$$

which proves the second part of the statement (2) in the claim. For the statement (3), just notice that the morphism ψ is an isomorphism at the generic point of Y . Thus we finish the proof of Claim 3.1.2.

Now twisting the short exact sequence $0 \longrightarrow I_{\tilde{V}} \longrightarrow \mathcal{O}_{\tilde{A}} \longrightarrow \mathcal{O}_{\tilde{V}} \longrightarrow 0$ by the divisor $\mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G)) \otimes \psi^* \omega_A$, we obtain an exact sequence

$$(3.1.3) \quad 0 \rightarrow I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G)) \otimes \psi^* \omega_A \rightarrow \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G)) \otimes \psi^* \omega_A \rightarrow \omega_{\tilde{V}} \rightarrow 0.$$

Push down this sequence via ψ . Notice that by the definition of multiplier ideal sheaves, we obtain

$$\psi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G)) = \mathcal{I}(A, cX).$$

Now we make the following claim.

Claim 3.1.4. We have the following statements for the sequence 3.1.3.

(1) Let $\mathcal{I}(A, cV)$ be the multiplier ideal sheaf associated to the pair (A, cV) , then

$$\psi_*(I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G))) = \mathcal{I}(A, cV).$$

(2) We have the vanishing

$$R^i \psi_*(I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G))) = 0, \quad \text{for } i > 0.$$

Proof of Claim 3.1.4. Let $\nu : A' \longrightarrow \tilde{A}$ be the blowing-up of \tilde{A} along \tilde{V} such that $I_{\tilde{V}} \cdot \mathcal{O}_{A'} = \mathcal{O}_{A'}(-S)$, where S is an exceptional divisor of ν . Notice that $K_{A'/\tilde{A}} = (c-1)S$ and $K_{A'/A} = K_{A'/\tilde{A}} + \nu^* K_{\tilde{A}/A}$. Thus we have

$$-S + \nu^* K_{\tilde{A}/A} - c\nu^*(T + \mu^* G) = K_{A'/A} - c(S + \nu^* T + (\mu \circ \nu)^* G).$$

Write the divisor $D := (S + \nu^* T + (\mu \circ \nu)^* G)$. We notice that $I_V \cdot \mathcal{O}_{A'} = \mathcal{O}_{A'}(-D)$. Since $\nu_* \mathcal{O}_{A'}(-S) = I_{\tilde{V}}$ and $R^i \nu_* \mathcal{O}_{A'}(-S) = 0$ for $i > 0$, we obtain that

$$\nu_* \mathcal{O}_{A'}(K_{A'/A} - cD) = I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G))$$

and

$$R^i \nu_* \mathcal{O}_{A'}(K_{A'/A} - cD) = 0, \quad \text{for } i > 0.$$

Write $f := \psi \circ \nu : A' \longrightarrow A$. The divisor $-D$ is f -nef and then by the relative Kodaira vanishing theorem we have

$$R^i f_* \mathcal{O}_{A'}(K_{A'/A} - cD) = 0, \quad \text{for } i > 0.$$

Also by the definition of multiplier ideal sheaves we have $f_* \mathcal{O}_{A'}(K_{A'/A} - cD) = \mathcal{I}(A, cV)$. Now by using the spectral sequence

$$E_2^{p,q} = R^p \psi_*(R^q \nu_* \mathcal{O}_{A'}(K_{A'/A} - cD)) \Rightarrow R^{p+q} f_* \mathcal{O}_{A'}(K_{A'/A} - cD),$$

we immediately have that

$$R^i \psi_*(I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G))) = 0, \quad \text{for } i > 0,$$

and

$$\psi_*(I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c(T + \mu^* G))) = \mathcal{I}(A, cV),$$

which finish the proof of Claim 3.1.4.

Finally, we push down the sequence (3.1.3) via ψ to obtain an exact sequence

$$0 \longrightarrow \mathcal{I}(A, cV) \otimes \omega_A \longrightarrow \mathcal{I}(A, cX) \otimes \omega_A \longrightarrow \omega_Y^{GR} \longrightarrow 0.$$

Thus by restricting to Y we see $\mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A \simeq \omega_Y^{GR}$, which proves the proposition. \square

Next we show that the isomorphism proved in Proposition 3.1 is canonical in the sense that it fits into the following commutative diagram 3.2.1. To this end, we need to track carefully the isomorphisms constructed in the proposition. Since canonical sheaves are only unique up to isomorphism we need to fix our canonical sheaves uniformly in arguments. There is a canonical canonical sheaf, namely sheaf of regular differential forms, developed by Kunz [KW88], offering a concrete way to do so. We follow the notation of Lipman [Lip84] to denote by $\tilde{\omega}$ the sheaf of regular differential forms.

Let Z be a reduced of pure dimension d scheme of finite type over k . Denote by \mathcal{K}_Z the locally constant sheaf of total quotient ring of \mathcal{O}_Z . Its ring of global sections is $K(Z) := \mathcal{K}_Z(Z)$ which is a product of residue fields of the generic points of Z . Let $\Omega_{Z/k}^d := \wedge^d \Omega_{Z/k}^1$ be the d -th exterior power of the sheaf of Kähler differential one-form. Let $\bar{\Omega}_{K(Z)/k}^d$ be the locally constant sheaf of meromorphic d -forms on Z so that its module of global sections is $\bar{\Omega}_{K(Z)/k}^d(Z) = \Omega_{K(Z)/k}^d$. The sheaf $\tilde{\omega}_Z$ of regular differential forms of degree d of Z is defined in [KW88, Section 3] and it is a subsheaf of $\bar{\Omega}_{Z/k}^d$. Now let $f : Z' \rightarrow Z$ be a resolution of singularities of Z so that f is isomorphic at the generic points of Z . Then pushdown the inclusion $\tilde{\omega}_{Z'} \hookrightarrow \bar{\Omega}_{K(Z')/k}^d$ via f we have $f_* \tilde{\omega}_{Z'} \hookrightarrow f_* \bar{\Omega}_{K(Z')/k}^d$. But since f is generically isomorphism we see that $K(Z') = K(Z)$ and $f_* \bar{\Omega}_{K(Z')/k}^d = \bar{\Omega}_{K(Z)/k}^d$. Thus $f_* \tilde{\omega}_{Z'}$ is naturally included in $\bar{\Omega}_{K(Z)/k}^d$ as a subsheaf. The trace map $\text{tr} : f_* \tilde{\omega}_{Z'} \hookrightarrow \tilde{\omega}_Z$ is then the natural inclusion as subsheaves of $\bar{\Omega}_{K(Z)/k}^d$.

Our idea is that the isomorphism of different canonical sheaf ω_Z can be compared to this canonical $\tilde{\omega}_Z$ and such comparison can be realized by automorphisms of the locally constant sheaf $\bar{\Omega}_{K(Z)/k}^d$.

Proposition 3.2. *With notation as in Proposition 3.1, the isomorphism $\omega_Y^{GR} \simeq \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A$ is canonical in the sense that it fits into the following commutative diagram*

$$(3.2.1) \quad \begin{array}{ccc} \omega_Y^{GR} & \xrightarrow{\simeq} & \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A \\ \text{tr} \downarrow & & \downarrow i \\ \omega_Y & \xrightarrow{\simeq} & I_X \cdot \mathcal{O}_Y \otimes \omega_A \end{array}$$

where the bottom isomorphism is given by 2.1.1, tr is the trace map, and the inclusion i is induced by the inclusion $\mathcal{I}(A, cX) \hookrightarrow \mathcal{I}_X$ (cf. Lemma 2.5).

Proof. Keep notation and construction as in the proof of Proposition 3.1. Assume that $d := \dim X$. We first make the adjunction isomorphism $\omega_V \simeq \omega_A \otimes_{\mathcal{O}_A} \mathcal{O}_V$ precisely by using regular differential forms mentioned above. Recall that V is a complete intersection defined

by $I_V = (\alpha_1, \dots, \alpha_c)$. Then it is clear that $I_V/I_V^2 = \oplus \mathcal{O}_V \bar{\alpha}_i$. Following notation of [Lip84, Section 13] we set a sheaf

$$\mathcal{H}_{V,A} := \mathcal{H}om_{\mathcal{O}_V}(\wedge^c I_V/I_V^2, \tilde{\omega}_A/I_V \tilde{\omega}_A),$$

which is torsion free since $\tilde{\omega}_A$ is locally free. Notice that $\mathcal{H}_{V,A} = \det N_{V/A} \otimes_{\mathcal{O}_A} \tilde{\omega}_A$ where $N_{V/A}$ is the normal sheaf of V in A . We have the following commutative diagram

$$(3.2.2) \quad \begin{array}{ccccc} \tilde{\omega}_V & \xrightarrow{\simeq} & \mathcal{H}_{V,A} & \xrightarrow{\simeq} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_V \\ \downarrow & & \downarrow i_V & & \downarrow j_V \\ \bar{\Omega}_{K(V)/k}^d & \xrightarrow{\simeq_{a_V}} & \mathcal{H}_{V,A} \otimes_{\mathcal{O}_V} \mathcal{K}_V & \xrightarrow{\simeq_{b_V}} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_V \end{array}$$

where the left commutative square follows from [HS97, Theorem 2.3] (see also [Lip84, Corollary 13.7]) and the right commutative square is a consequence of that the sheaves inside $\mathcal{H}om$ are all locally free and $\det N_{V/A} \simeq \mathcal{O}_V$. The morphism i_V and j_V are injective because $\mathcal{H}_{V,A}$ and $\tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_V$ are torsion free. Furthermore the vertical morphisms in the diagram can be thought of induced by tensoring with the natural inclusion $\mathcal{O}_V \hookrightarrow \mathcal{K}_V$. Thus $\mathcal{H}_{V,A}$ and $\tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_V$ are naturally as subsheaves of those locally constant sheaves at the bottom of the diagram. The adjunction isomorphism $\tilde{\omega}_V \simeq \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_V$ is then induced by $a_V^{-1} \circ b_V^{-1}$, i.e. $\tilde{\omega}_V = (a_V^{-1} \circ b_V^{-1})(\tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_V)$. The isomorphisms a_V and b_V can be described precisely and determined completely at each generic point of V . Assume v is one generic point of V with the residue field $k(v)$. The local ring $\mathcal{O}_{A,v}$ has a regular system of parameters $\alpha_1, \dots, \alpha_c, x_1, \dots, x_d$. Then locally at an open set of V containing only v the sheaf $\bar{\Omega}_{K(V)/k}^d = k(v)d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d$, the sheaf $\mathcal{H}_{V,A} \otimes \mathcal{K}_V = k(v)\xi$ where ξ maps $\bar{\alpha}_1 \wedge \dots \wedge \bar{\alpha}_c$ to $d\alpha_1 \wedge \dots \wedge d\alpha_c \wedge dx_1 \wedge \dots \wedge dx_d$ and the sheaf $\tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_V = k(v)d\alpha_1 \wedge \dots \wedge d\alpha_c \wedge dx_1 \wedge \dots \wedge dx_d$. Then on this open neighborhood of v the bottom line of the diagram 3.2.2 can be written as

$$k(v)d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d \longrightarrow k(v)\xi \longrightarrow k(v)d\alpha_1 \wedge \dots \wedge d\alpha_c \wedge dx_1 \wedge \dots \wedge dx_d.$$

The isomorphisms a_V and b_V are defined in the way that $a_V(d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d) = \xi$ and $b_V(\xi) = d\alpha_1 \wedge \dots \wedge d\alpha_c \wedge dx_1 \wedge \dots \wedge dx_d$.

Next we need to make the isomorphism $\omega_Y \simeq I_X \cdot \mathcal{O}_Y \otimes \omega_A$ clearly. Write the sheaf

$$\mathcal{H}_{Y,V} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \tilde{\omega}_V/I_{Y/V} \tilde{\omega}_V)$$

where $I_{Y/V} = I_Y \cdot \mathcal{O}_V$. It is a torsion free since $\tilde{\omega}_V$ is locally free. There is a fundamental local homomorphism (cf. [Lip84, 13.1])

$$h : \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_Y, \tilde{\omega}_V) \longrightarrow \mathcal{H}_{Y,V}$$

which in our case is induced by the natural quotient morphism $\tilde{\omega}_V \longrightarrow \tilde{\omega}_V/I_{Y/V} \tilde{\omega}_V$. We have the following commutative diagram

$$(3.2.3) \quad \begin{array}{ccccc} \tilde{\omega}_Y & \xrightarrow{\simeq} & \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_Y, \tilde{\omega}_V) & \longrightarrow & \tilde{\omega}_V \otimes_{\mathcal{O}_V} I_X \cdot \mathcal{O}_V \\ \downarrow & & \downarrow h & & \downarrow u_Y \\ & & \mathcal{H}_{Y,V} & \xrightarrow{\simeq} & \tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y \\ & & \downarrow i_Y & & \downarrow j_Y \\ \bar{\Omega}_{K(Y)/k}^d & \xrightarrow{\simeq_{a_Y}} & \mathcal{H}_{Y,V} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y & \xrightarrow{\simeq_{b_Y}} & \tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{K}_Y. \end{array}$$

The left hand side big square is commutative checked by definition directly. The right two small squares are commutative because $\tilde{\omega}_V$ is locally free and $\mathcal{H}\text{om}_{\mathcal{O}_V}(\mathcal{O}_Y, \mathcal{O}_V) = I_X \cdot \mathcal{O}_V$. The morphisms i_Y and $i_Y \circ h$ are injective since sheaves involved are all torsion free. Now let v be a generic point of Y which is also a generic point of V . Suppose that the local ring $\mathcal{O}_{A,v}$ has a regular system of parameters $\alpha_1, \dots, \alpha_c, x_1, \dots, x_d$. Then locally at an open set of Y containing only v we see that the sheaves $\bar{\Omega}_{K(Y)/k}^d = k(v)d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d$, $\mathcal{K}_{Y,V} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y = k(v)\xi_Y$ where ξ_Y maps 1 to $d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d$, and $\tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{K}_Y = k(v)d\bar{x}_1 \wedge \dots \wedge d\bar{x}_d$. Thus it is easy to check that the morphism $b_Y \circ a_Y$ is the identity. Hence in the locally constant sheaf $\bar{\Omega}_{K(Y)/k}^d = \tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{K}_Y$ the canonical sheaf $\tilde{\omega}_Y$ is exactly the sheaf $\tilde{\omega}_V \otimes_{\mathcal{O}_V} I_X \cdot \mathcal{O}_V$. Furthermore since Y is generic linked to X via V we see that $I_X \cdot \mathcal{O}_V = I_X \cdot \mathcal{O}_Y$.

Now tensoring the diagram (3.2.2) with \mathcal{O}_Y over \mathcal{O}_V . Notice that $\mathcal{K}_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y = \mathcal{K}_Y$ since $I_{Y/V} \cdot \mathcal{K}_V = \mathcal{K}_X$. And combining the diagram (3.2.3) together we have

$$(3.2.4) \quad \begin{array}{ccccc} \tilde{\omega}_Y & \xrightarrow{=} & \tilde{\omega}_V \otimes_{\mathcal{O}_V} I_X \cdot \mathcal{O}_V & \xrightarrow{\quad} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} I_X \cdot \mathcal{O}_V \\ \downarrow & & \downarrow u_Y & & \downarrow \\ & & \tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y & \xrightarrow{\quad} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{O}_Y \\ & & \downarrow j_Y & & \downarrow \\ \bar{\Omega}_{K(Y)/k}^d & \xrightarrow[b_Y \circ a_Y]{=} & \tilde{\omega}_V \otimes_{\mathcal{O}_V} \mathcal{K}_Y & \xrightarrow[(b_V \circ a_V) \otimes 1_Y]{\simeq} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y. \end{array}$$

The two horizontal morphisms on the top and right of the diagram can be thought of as the restriction of the right bottom morphism of locally constant sheaves on their subsheaves. Thus the sheaf $\tilde{\omega}_Y$ is the image of $I_X \otimes \tilde{\omega}_A$ under the following morphisms

$$(3.2.5) \quad I_X \otimes \tilde{\omega}_A \hookrightarrow \tilde{\omega}_A \longrightarrow \tilde{\omega}_A \otimes_{\mathcal{O}_Y} \longrightarrow \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_Y \xrightarrow{(b_V \circ a_V) \otimes 1_Y^{-1}} \bar{\Omega}_{K(Y)/k}^d$$

where except $(b_V \circ a_V) \otimes 1_Y^{-1}$ the rest morphisms are all natural ones.

Now we look at the canonical sheaf $\tilde{\omega}_{\tilde{V}}$. Denote by $E := T + \mu^*G$ an effective divisor on \tilde{A} . Recall that \tilde{V} is locally a complete intersection on \tilde{A} and $\wedge^c I_{\tilde{V}}/I_{\tilde{V}}^2 = \mathcal{O}_{\tilde{A}}(cE)$. The normal sheaf $N_{\tilde{V}/\tilde{A}} = \mathcal{O}_{\tilde{V}}(-cE)$. We define the sheaf

$$\mathcal{H}_{\tilde{V}, \tilde{A}} = \mathcal{H}\text{om}_{\mathcal{O}_{\tilde{V}}}(\wedge^c I_{\tilde{V}}/I_{\tilde{V}}^2, \tilde{\omega}_{\tilde{A}}/I_{\tilde{V}}\tilde{\omega}_{\tilde{A}})$$

which is torsion free since $\tilde{\omega}_{\tilde{A}}$ is free. Exactly as in the situation of V on A , we have the following commutative diagram

$$(3.2.6) \quad \begin{array}{ccccc} \tilde{\omega}_{\tilde{V}} & \xrightarrow{\simeq} & \mathcal{H}_{\tilde{V}, \tilde{A}} & \xrightarrow{\simeq} & \tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{A}}} \mathcal{O}_{\tilde{V}}(-cE) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{\Omega}_{K(\tilde{V})/k}^d & \xrightarrow[a_{\tilde{V}}]{\simeq} & \mathcal{H}_{\tilde{V}, \tilde{A}} \otimes_{\mathcal{O}_{\tilde{V}}} \mathcal{K}_{\tilde{V}} & \xrightarrow[b_{\tilde{V}}]{\simeq} & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_V \end{array}$$

Thus the sheaf $\tilde{\omega}_{\tilde{V}}$ is the image of the sheaf $\tilde{\omega}_{\tilde{A}}(-cE)$ under the morphisms

$$(3.2.7) \quad \tilde{\omega}_{\tilde{A}}(-cE) \hookrightarrow \tilde{\omega}_{\tilde{A}} \longrightarrow \tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{V}}} \longrightarrow \tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{A}}} \mathcal{K}_{\tilde{V}} \xrightarrow{(a_{\tilde{V}} \circ b_{\tilde{V}})^{-1}} \bar{\Omega}_{K(\tilde{V})/k}^d$$

where except of $(a_{\tilde{V}} \circ b_{\tilde{V}})^{-1}$ all morphisms are natural ones. Push down (3.2.7) via ψ . Notice that $\psi_*(\tilde{\omega}_{\tilde{A}}(-cE)) = \mathcal{S}(A, cX) \otimes \tilde{\omega}_A$ and $\psi_*\tilde{\omega}_{\tilde{A}} = \tilde{\omega}_A$. Also since the birational morphism ψ is an isomorphism over $A \setminus X$ so it is an isomorphism around generic points of Y and \tilde{V} and

therefore $\psi_*(\tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{A}}} \mathcal{K}_{\tilde{V}}) = \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_Y$ and $\psi_* \tilde{\Omega}_{K(\tilde{V})/k}^d = \tilde{\Omega}_{K(Y)/k}^d$ as locally constant sheaves.

Furthermore at one generic point v of Y which is identical to a generic point of \tilde{V} since ψ is an isomorphism around v , we can choose the same local equation of the local ring $\mathcal{O}_{A,v}$ and $\mathcal{O}_{\tilde{A},v}$, for instance, $\alpha_1, \dots, \alpha_c, x_1, \dots, x_d$. Then we can check that the morphism $(a_{\tilde{V}} \circ b_{\tilde{V}})^{-1}$ is the same as the morphism $(b_V \circ a_V) \otimes 1_Y^{-1}$. Thus we see that the sheaf $\psi_* \tilde{\omega}_{\tilde{V}}$ is the image of $\mathcal{S}(A, cX) \otimes \omega_A$ under the morphisms

$$(3.2.8) \quad \mathcal{S}(A, cX) \otimes \tilde{\omega}_A \hookrightarrow \tilde{\omega}_A \longrightarrow \psi_*(\tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{V}}} \mathcal{K}_{\tilde{V}}) \longrightarrow \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_Y \xrightarrow{(b_V \circ a_V) \otimes 1_Y^{-1}} \tilde{\Omega}_{K(Y)/k}^d.$$

The fact that $\mathcal{S}(A, cX) \otimes \tilde{\omega}_A$ is mapped surjectively to $\psi_* \tilde{\omega}_{\tilde{V}}$ is guaranteed by Claim 3.1.4 in the proof of Proposition 3.1.

Finally we compare (3.2.5) with (3.2.8). We have the following commutative diagram on \tilde{A}

$$(3.2.9) \quad \begin{array}{ccccc} \psi^* \tilde{\omega}_A & \longrightarrow & \psi^* \tilde{\omega}_A \otimes_{\mathcal{O}_{\tilde{V}}} \mathcal{K}_{\tilde{V}} & \longrightarrow & \psi^* \tilde{\omega}_A \otimes_{\mathcal{O}_{\tilde{A}}} \mathcal{K}_{\tilde{V}} \\ \downarrow & & \downarrow & & \parallel \\ \tilde{\omega}_{\tilde{A}} & \longrightarrow & \tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{V}}} \mathcal{K}_{\tilde{V}} & \longrightarrow & \tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{A}}} \mathcal{K}_{\tilde{V}} \end{array}$$

where the vertical morphisms are induced by the morphism $\psi^* \Omega_{A/k}^1 \longrightarrow \Omega_{\tilde{A}/k}^1$. Push down the diagram and notice that $\psi_* \tilde{\omega}_{\tilde{A}} = \tilde{\omega}_A$ then we have the commutative diagram

$$(3.2.10) \quad \begin{array}{ccccc} & & \tilde{\omega}_A \otimes \psi_* \mathcal{O}_{\tilde{V}} & \longrightarrow & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_Y \\ & \nearrow g & \downarrow & & \parallel \\ \tilde{\omega}_A & \longrightarrow & \psi_*(\tilde{\omega}_{\tilde{A}} \otimes_{\mathcal{O}_{\tilde{V}}} \mathcal{K}_{\tilde{V}}) & \longrightarrow & \tilde{\omega}_A \otimes_{\mathcal{O}_A} \mathcal{K}_Y \end{array}$$

Since $\tilde{\omega}_A \otimes \psi_* \mathcal{O}_{\tilde{V}}$ is naturally a \mathcal{O}_Y -module the morphism g then factors through $\tilde{\omega}_A \longrightarrow \tilde{\omega}_A \otimes \mathcal{O}_Y \longrightarrow \tilde{\omega}_A \otimes \psi_* \mathcal{O}_{\tilde{V}}$. Now it is clear that the proposition follows from (3.2.5) and (3.2.8). \square

Combining Proposition 3.1 and 3.2 together we now arrive at the first main theorem of this section as follows.

Theorem 3.3. *With notation as in Definition 2.3 let Y be a generic link to an affine pair (A_k, cX_k) . Then $\omega_Y^{GR} \simeq \mathcal{S}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A$, where $\mathcal{S}(A, cX)$ is the multiplier ideal sheaf associated to the pair (A, cX) , and this isomorphism fits into the commutative diagram (3.2.1).*

Corollary 3.4. *With notation as in Definition 2.3, let Y be a generic link to an affine pair (A_k, cX_k) . Then $\omega_Y^{GR} = \omega_Y$ if and only if $I_{X_k} = \mathcal{S}(A_k, cX_k)$, where $\mathcal{S}(A_k, cX_k)$ is the multiplier ideal sheaf associated to the pair (A_k, cX_k) .*

Proof. Let $\mathcal{S}(A, cX)$ be the multiplier ideal sheaf associated to the pair (A, cX) . By the commutative diagram (3.2.1) we have $\omega_Y^{GR} = \omega_Y$ if and only if $\mathcal{S}(A, cX) \cdot \mathcal{O}_Y = I_X \cdot \mathcal{O}_Y$ if and only if $I_X + I_Y = \mathcal{S}(A, cX) + I_Y$. Intersecting with I_X and noticing that $I_X \cap I_Y = I_V$ and $\mathcal{S}(A, cX) \subseteq I_X$, we see that $I_X + I_Y = \mathcal{S}(A, cX) + I_Y$ if and only if $I_X = \mathcal{S}(A, cX) + I_V$.

Now since the morphism $A \longrightarrow A_k$ is smooth, we then have $\mathcal{S}(A, cX) = \mathcal{S}(A_k, cX_k) \cdot \mathcal{O}_A$ by [Laz04b, 9.5.45]. Also notice that $I_X = I_{X_k} \cdot \mathcal{O}_A$ and the ring extension $\mathcal{O}_{A_k} \longrightarrow \mathcal{O}_A$ is faithfully flat. Thus intersecting with \mathcal{O}_{A_k} , we conclude that $I_X = \mathcal{S}(A, cX) + I_V$ if and only if $I_{X_k} = \mathcal{S}(A_k, cX_k)$. \square

Now we can easily deduce a criterion when a generic link has rational singularities. It turns out that multiplier ideal sheaves determine rational singularities of a generic link.

Corollary 3.5. *With notation as in Definition 2.3, let Y be a generic link to an affine pair (A_k, cX_k) . Then Y has rational singularities if and only if X_k is Cohen-Macaulay and $I_{X_k} = \mathcal{J}(A_k, cX_k)$, where $\mathcal{J}(A_k, cX_k)$ is the multiplier ideal sheaf associated to the pair (A_k, cX_k) .*

Proof. Y has rational singularities if and only if Y is Cohen-Macaulay and $\omega_Y^{GR} = \omega_Y$. Then the result is clear from above. \square

Corollary 3.6. *With notation as in Definition 2.3, let Y be a generic link to an affine pair (A_k, cX_k) . Suppose that the pair (A_k, cX_k) is log canonical and X_k is Cohen-Macaulay. Then Y has rational singularities.*

Proof. By Ein's Lemma, that (A_k, cX_k) is log canonical implies $I_{X_k} = \mathcal{J}(A_k, cI_{X_k})$. Then the result follows from above. \square

Remark 3.7. Now let us go back to Ulrich's observation mentioned in Introduction. Still with notation as in Definition 2.3 and suppose that X_k is a local complete intersection with rational singularities. Then by the inversion of adjunction [EM09] the pair (A_k, cX_k) is log canonical. Thus Corollary 3.5 says that a generic link Y of X_k has rational singularities.

One of the important invariants measuring the singularities of a pair (A, X) is the log canonical threshold. Let us briefly recall its definition here. Let $f : A' \rightarrow A$ be a log resolution of a pair (A, X) so that

$$K_{A'/A} = \sum_{i=1}^s k_i E_i \text{ and } f^{-1}(X) = \sum_{i=1}^s a_i E_i,$$

where the union of E_i 's and $\text{Exc}(f)$ is a simple normal crossing divisor. Then the log canonical threshold of (A, X) is defined to be

$$\text{lct}(A, X) := \min_i \left\{ \frac{k_i + 1}{a_i} \right\}.$$

The second main theorem of this section says that log canonical thresholds always increase in generic linkage. Thus in the sense of pairs, singularities of a generic link would not become worse.

Theorem 3.8. *With notation as in Definition 2.3 let (A_k, cX_k) be an affine pair and let Y be a generic link to X via a complete intersection V . Then*

$$\text{lct}(A, Y) \geq \text{lct}(A, V) = \text{lct}(A, X) = \text{lct}(A_k, X_k).$$

Proof. Keep notation and construction as in the proof of Proposition 3.1. First of all it is clear that $\text{lct}(A, X) = \text{lct}(A_k, X_k)$ and since $I_V \subseteq I_Y$ we have $\text{lct}(A, Y) \geq \text{lct}(A, V)$. Thus it suffices to show $\text{lct}(A, V) = \text{lct}(A, X)$.

Recall that $\varphi_k : \overline{A}_k \rightarrow A_k$ is a factorization resolution of singularities of X_k inside A_k and \overline{A}_k is the strict transformation of X_k . Denote by $\text{Exc}(\varphi_k) = \cup_{i=1}^s E_i^k$ the exceptional locus of φ_k where E_i^k are prime divisors with normal crossing support. Then \overline{A}_k has normal crossing with E_1^k, \dots, E_s^k (cf. Definition 2.6). We then can write the effective divisor $G_k = \sum_{i=1}^s a_i E_i^k$ and the relative canonical divisor $K_{\overline{A}_k/A_k} = \sum_{i=1}^s k_i E_i^k$.

Recall also that the factorizing resolution of singularities $\varphi : \overline{A} \rightarrow A$ of X inside A is then obtained by tensoring $\text{Spec } k[U_{ij}]$ with the resolution φ_k . Write $E_i := E_i^k \otimes_k \text{Spec } k[U_{ij}]$ for $i = 1, \dots, s$. Then it is clear that the exceptional locus of φ is $\text{Exc}(\varphi) = \cup_{i=1}^s E_i$ with

simple normal crossing support, and the strict transformation \overline{X} of X has normal crossing with E_1, \dots, E_s . Furthermore the effective divisor $G = \sum_{i=1}^s a_i E_i$ and the relative canonical divisor $K_{\overline{A}/A} = \sum_{i=1}^s k_i E_i$.

Now the morphism $\psi : \tilde{A} \rightarrow A$ is the composition of φ with the blowing-up $\mu : \tilde{A} \rightarrow A$ along \overline{X} with the exceptional divisor T . We set $\tilde{E}_i = \mu^*(E_i)$ for $i = 1, \dots, s$. Since \overline{X} has normal crossing with E_i we see that \tilde{E}_i is a prime divisor and $\text{Exc}(\psi) = T \cup_{i=1}^s \tilde{E}_i$ has simple normal crossing support. Thus $\psi : \tilde{A} \rightarrow A$ is a log resolution of X inside A . Notice that $K_{\tilde{A}/A} = (c-1)T$. We then can write

$$(3.8.1) \quad K_{\tilde{A}/A} = (c-1)T + \sum_{i=1}^s k_i \tilde{E}_i, \quad \psi^{-1}(X) = T + \sum_{i=1}^s a_i \tilde{E}_i.$$

Claim 3.8.2. Recall that \tilde{V} is nonsingular locally complete intersection on \tilde{A} (cf. Claim 3.1.2). Then \tilde{V} has normal crossing with $T, \tilde{E}_1, \dots, \tilde{E}_s$.

Proof of Claim 3.8.2. The question is local so we just need to look at local equations. Let $\overline{U}_k = \text{Spec } \overline{R}_k$ be an affine open set of \overline{A}_k such that $I_{\overline{X}_k} = (\overline{f}_1, \dots, \overline{f}_t) \subset \overline{R}_k$ and E_i^k has a local equation $h_i \in \overline{R}_k$ for $i = 1, \dots, s$ (cf. proof of Claim 3.1.1). Let $\overline{U} = \overline{U}_k \otimes \text{Spec } k[U_{ij}]$ be the corresponding affine open set in \overline{A} . Write $\overline{R} = \overline{R}_k \otimes k[U_{ij}]$. Then $I_{\overline{X}} = (\overline{f}_1, \dots, \overline{f}_t) \cdot \overline{R}$ and each E_i is still defined by the equation h_i in the ring \overline{R} . Now let $U_1 = \text{Spec } \overline{R}[\overline{f}_2/\overline{f}_1, \dots, \overline{f}_t/\overline{f}_1]$ be one canonical cover of \tilde{A} over U . Then on U_1 the divisor T is defined by the equation \overline{f}_1 . Notice that each \tilde{E}_i is still defined by the local equation $h_i \in \overline{R}[\overline{f}_2/\overline{f}_1, \dots, \overline{f}_t/\overline{f}_1]$. On U_1 the variety \tilde{V} is defined by $I_{\tilde{V}} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_c)$, where

$$\tilde{\alpha}_i = U_{i,1} + U_{i,2}\overline{f}_2/\overline{f}_1 + \dots + U_{i,t}\overline{f}_t/\overline{f}_1, \quad \text{for } i = 1, \dots, c.$$

Now we just need to show on U_1 , $I_{\tilde{V}}, \overline{f}_1, h_1, \dots, h_c$ are normal crossings. Notice that $\overline{f}_1, h_1, \dots, h_c$ are already normal crossings by the construction and they all sit in the ring \overline{R}_k . But $I_{\tilde{V}}$ is essentially defined by variables $\tilde{\alpha}_i$'s over \overline{R}_k . Thus a local calculation shows that $I_{\tilde{V}}$ meet $\overline{f}_1, h_1, \dots, h_c$ as normal crossings. This finishes the proof of Claim 3.8.2.

Now recall $\nu : A' \rightarrow A$ is the blowing-up of \tilde{A} along \tilde{V} with the exceptional divisor S and $f = \nu \circ \psi : A' \rightarrow A$. Write $E'_i = \nu^* \tilde{E}_i$ for $i = 1, \dots, s$ and $T' = \nu^* T$. By Claim 3.8.2 above we see that T', E'_1, \dots, E'_s are all prime divisors and $\text{Exc}(f) = S \cup T' \cup_{i=1}^s E'_i$ are simple normal crossings. Thus $f : A' \rightarrow A$ is a log resolution of $(A, V + X)$, which we use to compute log canonical thresholds. Notice that $K_{A'/A} = (c-1)S$. From (3.8.1) we can write

$$K_{A'/A} = (c-1)S + (c-1)T' + \sum_{i=1}^s k_i E'_i, \quad f^{-1}(X) = T' + \sum_{i=1}^s a_i E'_i.$$

Since $I_V \cdot \mathcal{O}_{\tilde{A}} = I_{\tilde{V}} \cdot \mathcal{O}_{\tilde{A}}(-\psi^{-1}(X))$ (cf. Claim 3.1.1 and 3.1.2) we then have

$$f^{-1}(V) = S + T' + \sum_{i=1}^s a_i E'_i.$$

Finally by the definition of log canonical threshold we see that

$$\text{lct}(A, X) = \min \left\{ \frac{k_i + 1}{a_i}, \frac{(c-1) + 1}{1}, \frac{(c-1) + 1}{0} \right\}$$

and

$$\text{lct}(A, V) = \min\left\{\frac{k_i + 1}{a_i}, \frac{(c-1) + 1}{1}, \frac{(c-1) + 1}{1}\right\}.$$

Therefore $\text{lct}(A, X) = \text{lct}(A, V)$ as required. \square

Corollary 3.9. *With notation as in Definition 2.3, if $I_X = \mathcal{J}(A, cX)$, where $\mathcal{J}(A, cX)$ is the multiplier ideal sheaf associated to the pair (A, cX) , then*

$$I_V = \mathcal{J}(A, cV) \text{ and } I_Y = \mathcal{J}(A, cY),$$

where $\mathcal{J}(A, cV)$ and $\mathcal{J}(A, cY)$ are multiplier ideal sheaves associated to the pairs (A, cV) and (A, cY) , respectively.

Proof. By Ein's Lemma, $I_X = \mathcal{J}(A, cX)$ if and only if $\mathcal{J}(A, (c-1)X) = \mathcal{O}_A$. Thus $\text{lct}(A, X) > (c-1)$ and therefore by Theorem 3.8 $\text{lct}(A, Y) \geq \text{lct}(A, V) = \text{lct}(A, X) > (c-1)$. Hence the multiplier ideal sheaves $\mathcal{J}(A, (c-1)Y)$ and $\mathcal{J}(A, (c-1)V)$ are all trivial. The result then follows by using Ein's Lemma again. \square

Remark 3.10. In the above corollary, the equality $\mathcal{J}_X = \mathcal{J}(A, cX)$ is equivalent to the equality $I_{X_k} = \mathcal{J}(A_k, cX_k)$, where $\mathcal{J}(A_k, cX_k)$ is the multiplier ideal sheaf associated to the pair (A_k, cX_k) . This is because the morphism $A \rightarrow A_k$ is smooth and the ring extension $R_k \rightarrow R$ is faithfully flat.

Corollary 3.11. *With notation as in Definition 2.3, let Y be a generic link to an affine pair (A_k, cX_k) . Suppose that the pair (A_k, cX_k) is log canonical. Then the pair (A, cY) is also log canonical.*

Proof. Since, by assumption, (A_k, cX_k) is log canonical and, by Lemma 2.5, $\mathcal{J}(A_k, cX_k) \subseteq I_{X_k}$ we see that $\text{lct}(A_k, X_k) = c$. Thus by Theorem 3.8, we have $\text{lct}(A, Y) \geq c$. But by Lemma 2.5, we have $\text{lct}(A, Y) \leq c$. Therefore $\text{lct}(A, Y) = c$ and thus (A, cY) is log canonical. \square

Remark 3.12. Using results established above, we then look at a sequence of generic linkages. Precisely, let (A_k, cX_k) be an affine pair as in Definition 2.3 and set $Y_0 := X_k$ and $A_0 := A_k$. We denote a generic link of Y_0 to be Y_1 , which is in a nonsingular ambient space A_1 . We can continue to construct a generic link of Y_1 as Y_2 in a nonsingular ambient space A_2 . Consequently, we get a sequence Y_0, Y_1, \dots , such that each Y_i is a generic link of Y_{i-1} and is in a nonsingular variety A_i . Now we list some interesting consequences from the above results as follows.

- (1) If $I_{Y_0} = \mathcal{J}(A_0, cY_0)$, then $I_{Y_i} = \mathcal{J}(A_i, cY_i)$, i.e. equality of multiplier ideal with ideal is preserved by generic linkages.
- (2) If (A_0, cY_0) is log canonical then (A_i, cY_i) is log canonical, i.e. log canonical pair is preserved by generic linkage.
- (3) If (A_0, cY_0) is log canonical and Y_0 is rational then (A_i, cY_i) is log canonical and Y_i is rational, i.e. log canonical plus rational is preserved by generic linkages.
- (4) $\text{lct}(A_0, Y_0) \leq \text{lct}(A_1, Y_1) \leq \dots \leq c$, i.e. log canonical thresholds increase in generic linkages but bounded above by c .

Notice that in (4) we get an increasing but bounded above sequence. Thus it must have a limit, which we denoted by $\text{lct}_\infty(A_0, Y_0)$. It would be very interesting to know if $\text{lct}_\infty(A_0, Y_0)$ is independent on the choice of generic lineage sequence Y_1, Y_2, \dots . It has been conjectured by the author that after finitely many steps of generic link sequences we will have $\text{lct}_\infty(A_0, Y_0) = c$. However, it was pointed out by Bernd Ulrich that the conjecture is not true because otherwise if we start with a Cohen-Macaulay Y_0 we will end up with a variety Y which has rational singularities, but it is not the case.

Remark 3.13. There is a conjecture made by the author in [Niu11, Conjecture 1.4] which asserts that if X is a local complete intersection with log canonical singularities then a generic link Y of X is also a local complete intersection with log canonical singularities. Now it is clear that this conjecture is false. One main reason is that Y cannot be a local complete intersection. However, Corollary 3.11 says that the log canonical pair is preserved by generic linkages and in the conjecture the pair (A, cX) is actually log canonical.

In the last of this section, we consider specialization problem stated in Situation B of Introduction. We shall have similar results as above but the argument is based on a different technique. The basic idea is that we turn X into an effective divisor by using resolution of singularities and then choose equations of a complete intersection V by Bertini's theorem.

Proposition 3.14. *Let $A = \text{Spec } R$ be a nonsingular affine variety over k and $X \subset A$ be a subscheme of A defined by an ideal $I_X = (f_1, \dots, f_t)$. Let $\mu : \bar{A} \rightarrow A$ be a log resolution of (A, X) such that $I_X \cdot \mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-E)$. Suppose that X is reduced equidimensional of codimension c . Then there exists a Zariski open set $U \subset \mathbb{A}_k^{c \times t}$ such that for any scalar matrix $a = (a_{i,j}) \in U$ the ideal $I_V := (\alpha_1, \dots, \alpha_c)$, where $\alpha_i = a_{i,1}f_1 + \dots + a_{i,t}f_t$, either equals I_X , or has a decomposition*

$$I_V \cdot \mathcal{O}_{\bar{A}} = I_{\bar{V}} \cdot \mathcal{O}_{\bar{A}}(-E)$$

where $I_{\bar{V}}$ is an ideal sheaf on \bar{A} defining a nonsingular equidimensional subscheme \bar{V} (possibly reducible) of codimension c . Furthermore \bar{V} has normal crossings with $\text{Exc}(\mu)$ and has no irreducible component contained in E .

Proof. We can assume that μ is an isomorphism over the open set $A \setminus X$. First of all, since X is generically a complete intersection, it is a standard result that there exists a Zariski open set $U_1 \subset \mathbb{A}_k^{c \times t}$ such that for any scalar matrix $a = (a_{i,j}) \in U$ the ideal $I_V := (\alpha_1, \dots, \alpha_c)$, where $\alpha_i = a_{i,1}f_1 + \dots + a_{i,t}f_t$, is a complete intersection and equals I_X at each generic point of X .

Now consider \mathcal{O}_X as a line bundle on X and let δ be the sub-linear system of $|\mathcal{O}_X|$ spanned by the sections f_1, \dots, f_t . By the definition of I_X , we have a surjective morphism

$$\oplus^t \mathcal{O}_{\bar{A}} \xrightarrow{(f_1, \dots, f_t)} I_X \rightarrow 0,$$

which means that X is the base locus of δ and induces a surjective morphism

$$(3.14.1) \quad \oplus^t \mathcal{O}_{\bar{A}} \xrightarrow{(\mu^* f_1, \dots, \mu^* f_t)} \mathcal{O}_{\bar{A}}(-E) \rightarrow 0.$$

Since μ is dominant the linear system δ can be identical to a sub-linear system $\mu^* \delta$ of $|\mathcal{O}_{\bar{A}}|$. The surjective morphism in (3.14.1) shows that E is the base locus of the system $\mu^* \delta$. By removing the component E from each section $\mu^* f_i$ we obtain section s_i of $\mathcal{O}_{\bar{A}}(-E)$, so that s_1, \dots, s_t span the sub-linear system $\bar{\delta} := \mu^* \delta - E$ of $|\mathcal{O}_{\bar{A}}(-E)|$. From the surjective morphism in (3.14.1) again, we then obtain a surjective morphism

$$(3.14.2) \quad \oplus^t \mathcal{O}_{\bar{A}}(E) \xrightarrow{(s_1, \dots, s_t)} \mathcal{O}_{\bar{A}} \rightarrow 0,$$

which means that the linear system $\bar{\delta}$ is base point free.

Thus by Bertini's theorem we can choose consecutively c general element $\theta_1, \dots, \theta_c$ of $\bar{\delta}$ and they define effective divisors D_1, \dots, D_c , respectively, such that: (1) each D_i is nonsingular and the intersection $\bar{V} := D_1 \cap \dots \cap D_c$ is either empty or nonsingular and equidimensional; Furthermore, if \bar{V} is nonempty, then (2) \bar{V} has no irreducible components contained in E and (3) \bar{V} has normal crossing with the exceptional locus $\text{Exc}(\mu)$. (1)–(3) are all direct consequences of Bertini's theorem since $\bar{\delta}$ is base point free. The meaning of general $\theta_1, \dots, \theta_c$ is that there

exists an Zariski open set $U_2 \subset \mathbb{A}_k^{c \times t}$ such that for any scalar matrix $a = (a_{i,j}) \in U$, sections $\theta_i := a_{i,1}s_1 + \cdots + a_{i,t}s_t$, for $i = 1, \dots, c$, satisfy (1)–(3).

Now take $U = U_1 \cap U_2$. Then for any general $\theta_1, \dots, \theta_c$ defined by a scalar matrix $a = (a_{i,j}) \in U$, we defined $I_V := (\alpha_1, \dots, \alpha_c)$, where $\alpha_i = a_{i,1}f_1 + \cdots + a_{i,t}f_t$. Notice that V is the base locus of the linear system spanned by $\alpha_1, \dots, \alpha_c$. It is then clear that by removing E component from the section $\mu^*\alpha_i$ we then get the section θ_i . If \overline{V} is empty then away from X , I_V defines nothing since μ is an isomorphism over $A \setminus X$. Thus since at the generic points of $I_V = I_X$ and $I_V \subset I_X$, we conclude that $I_V = I_X$. If \overline{V} is not empty, then away from X , I_V defines a nonsingular subscheme (in facet, it defines the link Y of X via V by $I_Y := (I_V : I_X)$), which is the same as the image of \overline{V} . It is then clear that $I_V \cdot \mathcal{O}_{\overline{A}} = I_{\overline{V}} \cdot \mathcal{O}_{\overline{A}}(-E)$. \square

Theorem 3.15. *Let $A = \text{Spec } R$ be a nonsingular affine variety over k and $X \subset A$ be a subscheme of A defined by an ideal $I_X = (f_1, \dots, f_t)$. Suppose that X is reduced equidimensional of codimension c and not a complete intersection in A . Then there is an Zariski open set $U \subset \mathbb{A}_k^{c \times t}$ such that for any scalar matrix $a = (a_{i,j}) \in U$ if the ideal $I_V := (\alpha_1, \dots, \alpha_c)$, where $\alpha_i = a_{i,1}f_1 + \cdots + a_{i,t}f_t$, and $I_Y := (I_V : I_X)$ then*

- (1) I_V defines a complete intersection V of A and I_Y defines a reduced subscheme Y of A such that Y is geometrically linked to X via V .
- (2) $\omega_Y^{GR} \simeq \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \mathcal{O}_A$ so that it fits into the following commutative diagram

$$\begin{array}{ccc} \omega_Y^{GR} & \xrightarrow{\simeq} & \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes \omega_A \\ \text{tr} \downarrow & & \downarrow i \\ \omega_Y & \xrightarrow{\simeq} & I_X \cdot \mathcal{O}_Y \otimes \omega_A \end{array}$$

where the bottom one is given by 2.1.1, tr is the trace map, and the inclusion i is induced by the inclusion $\mathcal{I}(A, cX) \hookrightarrow \mathcal{I}_X$ (cf. Lemma 2.5).

- (3) $\text{lct}(A, Y) \geq \text{lct}(A, V) = \text{lct}(A, X)$.

Proof. For (1), all we need is to show Y is reduced since the rest are all standard. To see this, with notion as in Proposition 3.14, the morphism $\mu : \overline{V} \rightarrow Y$ is a resolution of singularities of Y . Since we can take μ as an isomorphism over A/X , we see that Y is generically reduced. But Y does not have embedded components and therefore is reduced.

For (2) and (3), again with notation as in Proposition 3.14, since \overline{V} has normal crossing with the exceptional locus $\text{Exc}(\mu)$, then the results follow exactly as in the proofs of Proposition 3.1, Proposition 3.2 and Theorem 3.8. \square

Remark 3.16. With above theorem in hand, Corollary 3.6, 3.9 and 3.11 are still true in the context of Theorem 3.15. However, in Corollary 3.4 and 3.5 the “if” parts are still true but “only if” parts are not in general.

Remark 3.17. As in Remark 3.12 (4), we can look at an increasing sequence of log canonical thresholds $\text{lct}(A, Y_0) \leq \text{lct}(A, Y_1) \leq \cdots \leq c$, in which Y_i is a generic link of Y_{i-1} in the context of Theorem 3.15. It was pointed out by Lawrence Ein that by using ACC for log canonical thresholds [dFEM10], after finitely many steps the number lct becomes stable in the sequence, which we denoted by $\text{lct}_\infty(A, Y_0)$. We hope that this new invariant will have further application in linkage classes of varieties.

4. GENERIC LINKAGE OF PROJECTIVE VARIETIES

In this section, we study a generic link of a subvariety of \mathbb{P}^n in Situation C of Introduction. The main idea is inspired by the work of Betram, Ein and Lazarsfeld [BEL91] and similar to

Theorem 3.15. Thus we shall be brief in proofs. Throughout this section, we suppose that $A = \mathbb{P}^n$ is a projective space over k and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ is the hyperplane line bundle on \mathbb{P}^n .

Theorem 4.1. *Suppose that $X \subset A$ is a reduced equidimensional subscheme of codimension c scheme-theoretically defined by the t sections $s_i \in H^0(A, L^{d_i})$ with $d_1 \geq d_2 \geq \dots \geq d_t$. Take c general sections $\alpha_i \in H^0(A, \mathcal{I}_X \otimes L^{d_i})$ for $i = 1, \dots, c$ and let V be a complete intersection defined by the vanishing of $\alpha_1, \dots, \alpha_c$. Let Y be a subscheme of A defined by $\mathcal{I}_Y := (\mathcal{I}_V : \mathcal{I}_X)$. Then either Y is empty or else:*

- (1) Y is reduced equidimensional of codimension c (possibly reducible) geometrically linked to X via V .
- (2) $\omega_Y^{GR} \simeq \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes L^{d_1 + \dots + d_c} \otimes \omega_A$, where $\mathcal{I}(A, cX)$ is the multiplier ideal sheaf associated to the pair (A, cX) , and it fits into the following commutative diagram

$$\begin{array}{ccc} \omega_Y^{GR} & \xrightarrow{\simeq} & \mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes L^{d_1 + \dots + d_c} \omega_A \\ \text{tr} \downarrow & & \downarrow i \\ \omega_Y & \xrightarrow{\simeq} & I_X \cdot \mathcal{O}_Y \otimes L^{d_1 + \dots + d_c} \omega_A \end{array}$$

where the bottom one is given by 2.1.2, tr is the trace map, and the inclusion i is induced by the inclusion $\mathcal{I}(A, cX) \hookrightarrow \mathcal{I}_X$ (cf. Lemma 2.5).

- (3) $\text{lct}(A, Y) \geq \text{lct}(A, V) = \text{lct}(A, X)$.

Proof. Take a log resolution of singularities for the pair (A, X) as $f : \bar{A} \rightarrow A$ such that $\mathcal{I}_X \cdot \mathcal{O}_{\bar{A}} = \mathcal{O}_{\bar{A}}(-E)$ where E is an effective divisor and $\text{Exc}(f) \cup E$ is a divisor with simple normal crossing support. We may also assume that the morphism f is an isomorphism over the open set $A \setminus X$.

For $i = 1, \dots, t$ we denote by \mathfrak{b}_i the sub-linear system of $|L^{d_i}|$ determined by the vector space $H^0(A, \mathcal{I}_X \otimes L^{d_i})$. We use notation $(s_i)_0$ to be the effective divisor in the linear system \mathfrak{b}_i defined by the zero locus of the section s_i . Since X is defined by the vanishing of sections s_i we have a surjective morphism $\bigoplus^t L^{-d_i} \xrightarrow{(s_1, \dots, s_t)} \mathcal{I}_X \rightarrow 0$. Pulling back this surjective morphism via f , we then obtain a surjective morphism

$$(4.1.1) \quad \bigoplus^t f^* L^{-d_i} \xrightarrow{(f^* s_1, \dots, f^* s_t)} \mathcal{O}_{\bar{A}}(-E) \rightarrow 0.$$

(Since f is dominant the induced morphism on the linear system $f^* : \mathfrak{b}_i \rightarrow |f^* L^{d_i}|$ is actually injective. So we always think of \mathfrak{b}_i naturally as a sub-linear system of $|f^* L^{d_i}|$.) Denote by $\mathfrak{B}_i = f^* \mathfrak{b}_i - E$ the sub-linear system of $|f^* L^{d_i}(-E)|$ obtained from $f^* \mathfrak{b}_i$ by removing the base locus divisor E . Then the section $f^* s_i$ naturally gives rise to a section σ_i of $f^* L^{d_i}(-E)$ defining an effective divisor F_i in the linear system \mathfrak{B}_i . Thus from the surjectivity of (4.1.1), we deduce a surjection

$$\bigoplus^t f^* L^{-d_i}(E) \xrightarrow{(\sigma_1, \dots, \sigma_t)} \mathcal{O}_{\bar{A}} \rightarrow 0,$$

and therefore we have

$$(4.1.2) \quad F_1 \cap F_2 \cap \dots \cap F_t = \emptyset.$$

We make the following observation.

Claim 4.1.3. For each $i = 1, \dots, t$, one has

- (a) The system \mathfrak{B}_1 is base point free.

- (b) For each $i \geq 1$, the base locus $\text{Bs}(\mathfrak{B}_i)$ of the system \mathfrak{B}_i is inside the support of the divisor F_j for $j \geq i$.

Proof of Claim 4.1.3. For the statement (a), by the definition of X , we see that the sheaf $\mathcal{I}_X \otimes L^{d_1}$ is generated by its global sections. Let $W_1 = H^0(A, \mathcal{I}_X \otimes L^{d_1})$ so that $\mathfrak{b}_1 = |W|$. So we have a surjective morphism $W_1 \otimes L^{d_1} \rightarrow \mathcal{I}_X \rightarrow 0$. Thus we have a surjective morphism $W_1 \otimes f^*L^{-d_1} \rightarrow \mathcal{O}_{\overline{A}}(-E) \rightarrow 0$, i.e., $W_1 \otimes f^*L^{-d_1}(E) \rightarrow \mathcal{O}_{\overline{A}} \rightarrow 0$. Thus we see that \mathfrak{B}_1 is base point free.

For the statement (b), notice that when $i = 1$ the result is trivial from (a). We prove the first nontrivial case when $i = 2$. It is from the definition of base loci that $\text{Bs}(B_2) \subset F_2$. Now we show $\text{Bs}(B_2) \subset F_3$. Denote by δ_3^2 the linear system $|L^{d_2-d_3}|$ which is base point free. Notice that we have an inclusion $\delta_3^2 + (s_3)_0 \subset \mathfrak{b}_2$. Thus pulling back via f , we have the inclusion $f^*\delta_3^2 + f^*(s_3)_0 \subset f^*\mathfrak{b}_2$ and therefore by subtracting the divisor E we see $f^*\delta_3^2 + f^*(s_3)_0 - E \subset f^*\mathfrak{b}_2 - E$. Recall that $F_3 = f^*(s_3)_0 - E$ and $\mathfrak{B}_2 = f^*\mathfrak{b}_2 - E$. Thus we have an inclusion

$$f^*\delta_3^2 + F_3 \subset \mathfrak{B}_2.$$

From this, the linear system $\mathfrak{B}_2 - F_3$ contains the system $f^*\delta_3^2$, which is base point free. Thus the base locus $\text{Bs}(\mathfrak{B}_2)$ is contained in F_3 . Similar argument works for all $j \geq i$, which proves the Claim 4.1.3.

Now since \mathfrak{B}_1 is base point free, by Bertini's theorem (Cf. [Har77, Corollary III.10.9]) we can take a general element $D_1 \in \mathfrak{B}_1$ such that (i) D_1 is nonsingular and equidimensional; (ii) no components of D_1 are contained in the support of $E \cup \text{Exc}(f)$; (iii) D_1 has normal crossing with $\text{Exc}(f)$; (iv) $D_1 \cap F_2 \cap \cdots \cap F_t = \emptyset$. Here the reason for (iv) is that since the section σ_1 is nowhere vanishing on $F_2 \cap \cdots \cap F_t$, the general section of \mathfrak{B}_1 is then nowhere vanishing on $F_2 \cap \cdots \cap F_t$.

Now by Claim 4.1.3 the base locus $\text{Bs}(\mathfrak{B}_2)$ is inside F_j for $j \geq 2$. Thus we have $\text{Bs}(\mathfrak{B}_2) \subset F_2 \cap F_3 \cap \cdots \cap F_t$. By the fact (4.1.2) and the choice of D_1 , the linear system \mathfrak{B}_2 is base point free on D_1 . Thus by Bertini's theorem again we can choose a general element $D_2 \in \mathfrak{B}_2$ such that (i) $D_1 \cap D_2$ is nonsingular and equidimensional; (ii) no components of $D_1 \cap D_2$ are contained in the support of $E \cup \text{Exc}(f)$; (iii) $D_1 \cap D_2$ has normal crossing with $\text{Exc}(f)$; (iv) $D_1 \cap D_2 \cap F_3 \cap \cdots \cap F_t = \emptyset$.

We then can iterate such argument by c times to end up with a subscheme

$$\overline{Y} := D_1 \cap D_2 \cap \cdots \cap D_c$$

of \overline{A} such that \overline{Y} is either empty or else (i) \overline{Y} is nonsingular and equidimensional; (ii) no component of \overline{Y} is contained in the support of $E \cup \text{Exc}(f)$; (iii) \overline{Y} has normal crossing with $\text{Exc}(f)$; (iv) $\overline{Y} \cap F_{c+1} \cap \cdots \cap F_t = \emptyset$. Notice that each effective divisor D_i is a general element in the linear system \mathfrak{B}_i .

Now each D_i naturally corresponds to a general section $\alpha_i \in H^0(A, \mathcal{I}_X \otimes L^{d_i})$. (Recall that the divisor $D_i + E$ is in $f^*\mathfrak{b}_i$.) Those $\alpha_1, \dots, \alpha_c$ cut out a complete intersection V on A . Let Y be the subscheme of A linked to X via V , i.e., Y is defined by an ideal sheaf $\mathcal{I}_Y := (\mathcal{I}_V : \mathcal{I}_X)$. It is well-known that Y is equidimensional of codimension c without embedded components and with no common components with X . Notice that at least set-theoretically $Y = f(\overline{Y})$. Recall that the morphism f is an isomorphism over the open set $A \setminus X$. Thus by the construction of \overline{Y} we see that $\overline{Y} \cap f^{-1}(U)$ is isomorphic to $Y \cap U$. Therefore f is an isomorphism at the generic points of \overline{Y} and then \overline{Y} is the strict transform of Y . Thus Y is generically smooth and therefore is reduced. The rest of the statement (1) are all standard result, so we would not repeat here.

By the construction of \overline{Y} , we have a surjective morphism $\oplus^c f^* L^{-d_i}(E) \longrightarrow \mathcal{I}_{\overline{Y}} \longrightarrow 0$. Thus the normal bundle of \overline{Y} inside \overline{A} is $\mathcal{N}_{\overline{Y}/\overline{A}} = (f^* L_1^{d_1}(-E) \oplus \cdots \oplus f^* L_1^{d_c}(-E)) \otimes \mathcal{O}_{\overline{Y}}$ and therefore its determinant is $\det \mathcal{N}_{\overline{Y}/\overline{A}} = f^* L^{d_1 + \cdots + d_c}(-cE) \otimes \mathcal{O}_{\overline{Y}}$. We denote by $d_X := d_1 + \cdots + d_c$. Then by the adjunction formula, we have

$$\omega_{\overline{Y}} \simeq \omega_{\overline{A}} \otimes \det \mathcal{N}_{\overline{Y}/\overline{A}} \simeq \mathcal{O}_{\overline{Y}}(K_{\overline{A}/A} - cE) \otimes f^* L^{d_X} \otimes f^* \omega_A,$$

where $K_{\overline{A}/A}$ is the relative canonical divisor of the morphism f . Now from an exact sequence $0 \longrightarrow \mathcal{I}_{\overline{Y}} \longrightarrow \mathcal{O}_{\overline{A}} \longrightarrow \mathcal{O}_{\overline{Y}} \longrightarrow 0$ we obtain

$$0 \rightarrow \mathcal{I}_{\overline{Y}} \cdot \mathcal{O}_{\overline{A}}(K_{\overline{A}/A} - cE) \otimes f^* L^{d_X} \otimes f^* \omega_A \rightarrow \mathcal{O}_{\overline{A}}(K_{\overline{A}/A} - cE) \otimes f^* L^{d_X} \otimes f^* \omega_A \rightarrow \omega_{\overline{Y}} \rightarrow 0.$$

Pushing down this sequence by f and noticing that $R^i f_* \mathcal{I}_{\overline{Y}} \cdot \mathcal{O}_{\overline{A}}(K_{\overline{A}/A} - cE) = 0$ for $i > 0$ as in the proof of Claim 3.1.4 in the previous section, we then obtain an exact sequence

$$0 \longrightarrow \mathcal{I}(A, cV) \otimes L^{d_X} \otimes \omega_A \longrightarrow \mathcal{I}(A, cX) \otimes L^{d_X} \otimes \omega_A \longrightarrow \omega_Y^{GR} \longrightarrow 0.$$

By restricting to Y , we then have $\mathcal{I}(A, cX) \cdot \mathcal{O}_Y \otimes L^{d_X} \otimes \omega_A \simeq \omega_Y^{GR}$. The commutative diagram in the statement (2) can be checked as in Proposition 3.2.

Finally, the statement (3) has the same proof as in Theorem 3.8 so we would not repeat here. \square

In the theorem above, a generic link Y is usually not necessarily irreducible. One important case in applications is when X is cut out by equations of the same degree, i.e., $d_1 = \cdots = d_t$. Furthermore, if the defining ideal sheaf \mathcal{I}_X has enough sections then it is possible that a generic link Y is irreducible. One way to see this is by using the s -invariant of \mathcal{I}_X with respect to L , which measures the positivity of \mathcal{I}_X . We recall its definition and refer to [Laz04a] for further reference.

Definition 4.2. Given an ideal sheaf \mathcal{I} on \mathbb{P}^n let $\mu : W = \text{Bl}_{\mathcal{I}} \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be the blowing-up of \mathbb{P}^n along the ideal \mathcal{I} with an exceptional Cartier divisor E on W , such that $\mathcal{I} \cdot \mathcal{O}_W = \mathcal{O}_W(-E)$. Let H be the hyperplane divisor of \mathbb{P}^n . We define the s -invariant of \mathcal{I} with respect to H to be the positive real number

$$s(\mathcal{I}) := \min \{ s \mid s\mu^*H - E \text{ is nef} \}.$$

Here $s\mu^*H - E$ is considered as an \mathbb{R} -divisor on W .

Remark 4.3. Suppose that d is an integer such that $\mathcal{I}(d)$ is generated by global sections, then it is easy to see that $s(\mathcal{I}) \leq d$, i.e., the s -invariant of \mathcal{I} is always bounded by a generating degree of \mathcal{I} . There is an example in [CEL01] showing that the s -invariant could be an irrational number.

Corollary 4.4. *With notation and assumption as in Theorem 4.1 assume further that X is cut out by the sections of the same degree, i.e., $d_1 = \cdots = d_t = d$. If the s -invariant of \mathcal{I}_X with respect to L is strictly smaller than d , then Y is nonempty and irreducible.*

Proof. We keep notation as in the proof of Theorem 4.1 but set $d_1 = \cdots = d_t = d$. Now this time the linear systems $\mathfrak{B}_1, \dots, \mathfrak{B}_t$ are all the same as the linear system $\mathfrak{B} := f^* \mathfrak{b} - E$, where \mathfrak{b} is the sub-linear system of $|L^d|$ determined by the vector space $H^0(A, \mathcal{I}_X \otimes L^d)$. Notice that \mathfrak{B} is a sub-linear system of $|f^* L^d(-E)|$ and is base point free. Also notice that $\dim \mathfrak{B} = \dim \mathfrak{b}$ since f is dominant and E is the basic locus of $f^* \mathfrak{b}$.

All we need is to show the subscheme \overline{Y} , which is the intersection of c general divisors $D_i \in \mathfrak{B}$, is irreducible. Recall that \mathfrak{B} is base point free, it then gives a morphism to a projective space

$$\phi_{\mathfrak{B}} : \overline{A} \longrightarrow \mathbb{P}^r,$$

where $r = \dim \mathfrak{B}$ such that $f^*L^d(-E) = \phi_{\mathfrak{B}}^*\mathcal{O}_{\mathbb{P}^r}(1)$. We claim that the morphism $\phi_{\mathfrak{B}}$ is generically finite. To see this, let $\mu : A' = \text{Bl}_X A \longrightarrow A$ be the blowing up of A along X with an exceptional divisor F such that $\mathcal{I}_X \cdot \mathcal{O}_{A'} = \mathcal{O}_{A'}(-F)$. By the universal property of blowing-ups we have $g : \overline{A} \longrightarrow A'$ such that $f = g \circ \mu$ and $g^*F = E$. Notice that g is generically finite. Now the system $\mathfrak{B}' := \mu^*\mathfrak{b} - F$ is base point free and then gives a morphism $\phi_{\mathfrak{B}'}$ to \mathbb{P}^r , which commutes with $\phi_{\mathfrak{B}}$, i.e. $\phi_{\mathfrak{B}} = \phi_{\mathfrak{B}'} \circ g$

$$\begin{array}{ccc} \overline{A} & \xrightarrow{\phi_{\mathfrak{B}}} & \mathbb{P}^r \\ & \searrow g & \nearrow \phi_{\mathfrak{B}'} \\ & A' & \end{array}$$

Since the s -invariant of \mathcal{I}_X with respect to L is strictly smaller than d the line bundle $\mu^*L^d(-F)$ is ample. But $\mu^*L^d(-F) = \phi_{\mathfrak{B}'}^*\mathcal{O}_{\mathbb{P}^r}(1)$ so the morphism $\phi_{\mathfrak{B}'}$ is finite. Thus $\phi_{\mathfrak{B}}$ is generically finite and we have $\dim \phi_{\mathfrak{B}}(\overline{A}) = \dim \overline{A}$. Now by the theorem of [Laz04a, 3.3.1], the subscheme \overline{Y} is nonempty and irreducible since $\dim \phi_{\mathfrak{B}}(\overline{A}) > c$. Therefore the generic link Y to X is also nonempty and irreducible. \square

Remark 4.5. Having Theorem 4.1 in hand, it is then easy to get those similar corollaries mentioned in the previous sections (cf. Remark 3.12, 3.16 and 3.17). So we leave them to the reader.

REFERENCES

- [BEL91] Aaron Bertram, Lawrence Ein, and Robert Lazarsfeld. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.*, 4(3):587–602, 1991.
- [BVU03] A. Bravo and O. Villamayor U. A strengthening of resolution of singularities in characteristic zero. *Proc. London Math. Soc. (3)*, 86(2):327–357, 2003.
- [CEL01] Steven Dale Cutkosky, Lawrence Ein, and Robert Lazarsfeld. Positivity and complexity of ideal sheaves. *Math. Ann.*, 321(2):213–234, 2001.
- [dFD12] Tommaso de Fernex and Roi Docampo. Jacobian discrepancies and rational singularities. 2012.
- [dFEM10] Tommaso de Fernex, Lawrence Ein, and Mircea Mustaă. Shokurov’s ACC conjecture for log canonical thresholds on smooth varieties. *Duke Math. J.*, 152(1):93–114, 2010.
- [EIM11] Lawrence Ein, Shihoko Ishii, and Mircea Mustata. *Multiplier ideals via Mather discrepancy*. Arxiv, 2011.
- [EM09] Lawrence Ein and Mircea Mustaă. Jet schemes and singularities. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 505–546. Amer. Math. Soc., Providence, RI, 2009.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Hoc73] M. Hochster. Properties of Noetherian rings stable under general grade reduction. *Arch. Math. (Basel)*, 24:393–396, 1973.
- [HS97] Reinhold Hubl and Gerhard Seibert. The adjunction morphism for regular differential forms and relative duality. *Compositio Math.*, 106(1):87–123, 1997.
- [HU85] Craig Huneke and Bernd Ulrich. Divisor class groups and deformations. *Amer. J. Math.*, 107(6):1265–1303 (1986), 1985.
- [HU87] Craig Huneke and Bernd Ulrich. The structure of linkage. *Ann. of Math. (2)*, 126(2):277–334, 1987.
- [HU88] Craig Huneke and Bernd Ulrich. Algebraic linkage. *Duke Math. J.*, 56(3):415–429, 1988.
- [Kol97] Jnos Kollr. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.

- [KW88] Ernst Kunz and Rolf Waldi. *Regular differential forms*, volume 79 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1988.
- [Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.
- [Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
- [Lip84] Joseph Lipman. Dualizing sheaves, differentials and residues on algebraic varieties. *Astérisque*, (117):ii+138, 1984.
- [Mig98] Juan C. Migliore. *Introduction to liaison theory and deficiency modules*, volume 165 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1998.
- [Niu11] Wenbo Niu. A bound for the Castelnuovo-Mumford regularity of log canonical varieties. *J. Pure Appl. Algebra*, 215(9):2180–2189, 2011.
- [PS74] C. Peskine and L. Szpiro. Liaison des variétés algébriques. I. *Invent. Math.*, 26:271–302, 1974.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907-2067, USA

E-mail address: niu6@math.purdue.edu